

## Dynamics of Fractal Structures

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Dilation symmetry, as opposed to the degenerate limit of translation symmetry, requires (at least) three dimensionalities to contain a physical description:  $d$ , the Euclidean (or embedding) dimension;  $\bar{d}$ , the Hausdorff (or fractal) dimension;  $\bar{\bar{d}}$ , the fracton (or spectral) dimension. The dynamical properties of percolating networks are examined in this context. The vibrational density of states,  $N(\omega)$ , is calculated, and shown to be proportional to  $\omega^{d-1}$  in the phonon, or long-length-scale regime. A crossover is found at a frequency  $\omega_c$ , proportional to  $p - p_c$ , where  $p$  is the bond occupancy probability, and  $p_c$  the critical percolation concentration. At short length scales,  $N(\omega)$  is proportional to  $\omega^{\bar{d}-1}$ , and the excitations are termed *fractons*. An effective medium approximation (EMA) calculation of the vibration density of states exhibits a rapid rise in  $N(\omega)$  in the vicinity of  $\omega_c$ . We suggest that this overall behavior has relevance to the vibrational properties of amorphous materials. The far infrared absorption spectra of a number of glasses and amorphous Ge exhibit structures which appear similar to the calculated EMA  $N(\omega)$ . This lends credence to our previous analysis based solely on the thermal properties. We use the EMA to compute  $\langle r^2(t) \rangle$  for a percolating network, and thence calculate the diffusion constant  $D(t)$ . For short times, we obtain the Webman EMA result,  $D(t) \propto t^{-1/2}$ , with a smooth crossover to a constant value for long times. The vibrational dispersion curves are calculated within EMA. The velocity of sound  $v_s$  is found to vary as  $(p - p_c)^{1/2}$  in the phonon (small wave vector  $q$ ) regime. When  $q \sim q_c$ , ( $q_c v_s = \omega$ ) for  $p$  near  $p_c$ , the dispersion curves flatten and bend over, then rise again with  $\omega \propto q^2$ , looking somewhat "roton"-like. For  $q > q_c$ , the "damping" becomes very large, so that the plane wave character of the solution fails. This peculiar double-valued structure in  $\omega(q)$  is responsible for the rapid rise in  $N(\omega)$  near  $\omega_c$ , and not the behavior of the diffusion constant. Our results suggest the following EMA values at  $d = 3$  in the fracton regime:  $\bar{d} = 1$ ,  $\bar{\bar{d}} = 2$ , or  $\theta = 2$  where  $D(r) \propto r^{-\theta}$ .

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## 1. INTRODUCTION

It is now generally recognized that self-similar fractal structures<sup>(1)</sup> require (at least) three dimensionalities for their description: the embedding Euclidean dimension  $d$ , the Hausdorff (fractal<sup>(1)</sup>) dimension  $\bar{d}$ , and the fracton<sup>(2)</sup> (spectral) dimension  $\bar{\bar{d}}$ . The latter two quantities collapse to the first as one passes from the self-similar regime which exhibits dilation symmetry<sup>(3)</sup> to the Euclidean regime where the system is translationally symmetric. The percolating network is an ideal system to study this "crossover" between the two very different regimes. A characteristic length scale is set by the percolation correlation length  $\xi_p \propto (p - p_c)^{-\nu}$  where  $p$  is the site or bond concentration, and  $p_c$  the critical percolation concentration. The exponent  $\nu = 4/3$  exactly<sup>(4)</sup> at  $d = 2$ , and is approximately 0.84 at  $d = 3$ .<sup>(5)</sup> For length scales longer than  $\xi_p$ , the system is Euclidean, while for length scales shorter than  $\xi_p$ , it is a self-similar fractal structure.<sup>(6)</sup>

Many physical systems are thought to map onto a percolating network. For example, de Gennes<sup>(7)</sup> uses the mapping for a description of gels. We have claimed<sup>(8)</sup> that the vibrational properties of cross-linked polymers, glasses, and irradiated quartz exhibit properties which are not dissimilar from those calculated for the percolating network. We based this claim originally on a scaling argument. The purpose of this paper is to use our recent effective medium approximation (EMA) calculations of the vibrational properties of the percolating network as a model for amorphous materials. These calculations show that the low-frequency (long-length-scale) vibrational excitations are phonon like with a density of states proportional to  $\omega^{d-1}$ . A crossover frequency exists (itself proportional to  $p - p_c$ ),  $\omega_c$ , corresponding to the length scale  $\xi_p$ . As  $\omega$  increases through  $\omega_c$ , a rapid rise in  $N(\omega)$  takes place, with  $N(\omega)$  then "saturating" to a constant value (for  $2 \leq d \leq 4$ ), corresponding to the fraction regime where<sup>(2)</sup>  $N(\omega) \propto \omega^{\bar{\bar{d}}-1}$  (within EMA,  $\bar{\bar{d}} = 1$  for  $2 \leq d \leq 4$ ). This shape for  $N(\omega)$  is similar to that extracted from specific heat measurements on epoxy-resin,<sup>(9)</sup> and from far infrared absorption measurements on amorphous  $\text{GeO}_2$ ,  $\text{B}_2\text{O}_3$ , and  $\text{SiO}_2$ , and on<sup>(11)</sup>  $\alpha$ -Ge.<sup>(10)</sup>

We explore the physical reasons why  $N(\omega)$  behaves in this manner by first calculating  $D(t)$ , the time-dependent diffusion constant, within EMA. We show that  $D(t)$  crosses over smoothly from the fractal (short-time) regime result of Webman,<sup>(12)</sup>  $D(t) \propto t^{-1/2}$ , to a constant in the Euclidean (long-time) regime. The unusual shape [the rapid rise in  $N(\omega)$ ] at  $\omega_c$  cannot then be blamed on quantities related to the diffusion constant (e.g., the effective bulk modulus<sup>(13,14)</sup>) but rather must have a different origin. We report the results of an EMA calculation of the dispersion law for vibrations on a percolating network. We find a phononlike behavior at long length

scales,  $\omega = v_s q$  where the velocity of sound  $v_s \propto p - p_c$ . For large  $q$ , such that  $\omega \sim \omega_c$  and  $p \sim p_c$ , the dispersion curve “rolls-over” to a minimum and then rises rapidly, with  $\omega \propto q^2$  (scaling theory<sup>(2,15)</sup> yields  $\omega \propto q^{\bar{d}/\bar{d}}$ , so that  $\bar{d} = 2$  at  $d = 3$  within the EMA). This roll-over is responsible for the rapid rise in  $N(\omega)$  near  $\omega_c$  for  $p \sim p_c$ . The dispersion solutions are overdamped beyond the first maximum in  $\omega(q)$ , signaling the failure of the plane wave solution for  $\omega > \omega_c$ . The “fracton” excitations which characterize this regime are thought to be localized.<sup>(3)</sup> We have argued<sup>(8)</sup> that the plateau observed in the thermal conductivity of amorphous materials<sup>(9,16)</sup> can be understood on this basis.

This paper sketches the derivation of the fraction dimensionality  $\bar{d}$  in Section 2, along with a scaling derivation of  $N(\omega)$  for lattice vibrations on a fractal network. Section 3 is concerned with the EMA treatment of  $N(\omega)$ , as well as the calculation of  $\langle r^2(t) \rangle$  and thence  $D(t)$ . Section 4 reviews the dispersion relation for lattice vibrations of a percolating network, and discusses the character of the solutions. Section 5 summarizes the paper, and suggests directions for future investigations.

## 2. SCALING DERIVATIONS OF $\bar{d}$ AND $N(\omega)$

The scaling theory for the diffusion constant on the percolating network was first fully developed by Gefen, Aharony, and Alexander.<sup>(17)</sup> They showed that for the problem of diffusion on random networks (“the unbiased ant in the labyrinth”<sup>(18-21)</sup>), for a single percolation cluster at short times,

$$\langle r^2(t) \rangle_s \propto t^{2/(2+\theta)} \tag{1}$$

Here, the cluster contains  $s$  sites within a linear range  $r_s \propto s^{1/\bar{d}}$ , and if  $a$  is the bond length,  $a \ll r(t) \ll (r_s, \xi_p)$ . This implies for the infinite cluster a diffusion constant range dependence:

$$D(r) \propto r^{-\theta} \tag{2}$$

Following the arguments in Ref. 2, the volume visited on the infinite cluster in time  $t$ ,  $\langle r^2(t) \rangle^{\bar{d}/2}$ , can be written

$$V(t) \propto t^{\bar{d}/(2+\theta)} \tag{3}$$

This means the probability of remaining at the initial ( $t = 0$ ) site after time  $t$  is

$$\langle P_0(t) \rangle \propto [V(t)]^{-1} \propto t^{-\bar{d}/(2+\theta)} \tag{4}$$

The structure of the diffusion equation is such that it can be mapped onto a master equation, which in turn has the same form as the free particle Schrödinger equation, the equation of motion for mechanical vibrations, and the linearized equation of motion for ferromagnetic spin deviations. This enables us to map the eigenvalue density of states for the quantum vibrational problem onto the eigenvalue density of states of the diffusion problem. The latter can be obtained from the single site Green's function for the diffusion problem,<sup>(22)</sup>

$$N(\varepsilon) = -(1/\pi) \text{Im} \langle \tilde{P}_0(-\varepsilon + i0^+) \rangle \quad (5)$$

where  $\tilde{P}_0(\varepsilon)$  is the Laplace transform of  $P_0(t)$ , the autocorrelation function, with  $\varepsilon$  the spectral parameter.

We find immediately that  $N(\varepsilon) \propto \varepsilon^x$ , with  $x = [\bar{d}/(2 + \theta)] - 1$ . The spectral parameter  $\varepsilon$  can be related to the energy eigenvalues of the vibrational problem by replacing  $\varepsilon$  by  $\omega^2$  and multiplying by  $\omega$ :

$$N(\omega) \propto \omega^{[2\bar{d}/(2+\theta)]-1} \quad (6)$$

Noting that the form for  $N(\omega)$  in Euclidean space is  $\omega^{d-1}$ , Alexander and Orbach<sup>(2)</sup> were led to identify the fracton dimensionality

$$\bar{\bar{d}} = 2\bar{d}/(2 + \theta) \quad (7)$$

They then expressed  $\bar{\bar{d}}$  in terms of  $\bar{d} = d - (\beta/v)$  and  $\theta = (\mu - \beta)/v$ , where the probability of being on the infinite cluster  $P(p) \propto (p - p_c)^\beta$ , and the dc conductivity  $\sigma \propto (p - p_c)^\mu$ . The (then) best values<sup>(5)</sup> for  $v$ ,  $\beta$ , and  $\mu$  led to values of  $\bar{\bar{d}} \cong 4/3$ , independent of  $d$  (for  $d \geq 2$ ). Alexander and Orbach conjectured that this relation might be exact, leading to an expression for  $\mu$  in terms of  $d$ ,  $v$ , and  $\beta$ :

$$\mu = \frac{1}{2}[v(3d - 4) - \beta] \quad (8)$$

or, from the den Nijs conjecture<sup>(4)</sup>  $v = 4/3$ , and  $\beta = 5/36$ , in  $d = 2$ ,

$$\mu/v = 91/96 = 0.9479... \quad (9)$$

The "superuniversal" relationship,<sup>(23)</sup>  $\bar{\bar{d}} = 4/3$ ,  $d \geq 2$ , has been challenged for  $d = 2$  in a very recent preprint by Zabolitzky,<sup>(24)</sup> who calculated the conductivity of a random network of resistors and insulators in two dimensions for strips of size  $N \times L$ , with  $L$  of the order of several  $10^6$ , and  $N$  up to 350. He finds  $\mu/v = 0.973 \pm 0.005$ . A similar value [ $s/v = 0.977 \pm 0.010$ ] has been found for a mixture of normal and superconducting bonds in  $d = 2$  [ $\mu = s$  in  $d = 2$ ] by Derrida, Herrman, and Vannimenus.<sup>(25)</sup> These results can be used

in reverse from Eq. (8) to argue that  $\bar{d} = 1.322 \pm 0.003$  and  $1.320 \pm 0.005$ , respectively. A recent paper by Aharony and Stauffer<sup>(26)</sup> suggests that  $\mu/\nu = \bar{d} - \beta/\nu = d - 1$  below the Euclidean dimensionality  $d$  such that  $\bar{d} = 2$ . They argue therefore that  $\mu/\nu = 1$  for  $d = 2$ , suggesting that  $\bar{d} = 4/3$  only above approximately  $d \approx 2.1$ . They suggest that Ref. 24 had not executed sufficient Monte Carlo steps to reach the asymptotic behavior  $\mu/\nu = 1$  for  $d = 2$ .

On the opposing side, Argyrakis and Kopelman<sup>(27)</sup> have performed random walk simulations at and above the percolation threshold for two- and three-dimensional lattices. They used the Rammal and Toulouse<sup>(3)</sup> suggestion that, at the percolation threshold, the number  $S_N$  of distinct sites visited during an  $N$ -step random walk on an infinite cluster varies asymptotically as

$$S_N \propto N^{\bar{d}/2} \quad (10)$$

For the square and cubic lattice ( $d = 2$  and  $3$ ), they report  $\bar{d} = 1.331$ , in apparent contradiction with the simulation values found for  $\mu/\nu$  of Refs. 24 and 25.

Geometrical arguments have been advanced for  $\bar{d} = 4/3$  by Rammal and Toulouse<sup>(3)</sup> and Leyvraz and Stanley.<sup>(28)</sup> The compactness of the visited sites<sup>(3)</sup> for low dimensions ( $\bar{d} \leq 2$ ) led Aharony and Stauffer<sup>(26)</sup> to a geometrical justification for  $\mu/\nu = 1$  (or  $\bar{d} = 1.309$  from scaling) at  $\bar{d} \leq 2$ . For  $d = 3$ , they agree with the conclusions of Leyvraz and Stanley.<sup>(28)</sup> The reader is referred to Refs. 26 and 28 for the detailed geometrical arguments.

From our perspective, there is at least one unanswered question: why do the strip simulations<sup>(24,25)</sup> in  $d = 2$  show a clear crossover to a value of  $\mu/\nu$  larger than Eq. (9), leading to  $\bar{d} < 4/3$  [in the Aharony–Stauffer<sup>(26)</sup> limit,  $\mu/\nu = 1$ ,  $\bar{d} = 1.309$ ] while the diffusion results in  $d = 2$  for the number of distinct sites visited of Ref. 27 lead to  $\bar{d}$  nearly exactly  $4/3$ ? We must await further numerical treatments for a resolution of this difficulty.

The principal result of this Section is Eq. (6) with the definition of  $\bar{d}$  in Eq. (7). It is shown in Ref. 15 that this form for the fracton density of states (incidentally, with  $\bar{d} = 4/3$ ) fits the higher-temperature portions of the specific heat of the epoxy-resins.<sup>(9)</sup>

### 3. EMA CALCULATION OF $N(\omega)$

The principal difficulty with the scaling results of the previous section is that they do not provide us with the insight necessary to connect the lower-frequency vibrational density of states for the phonons,  $N_{\text{ph}}(\omega) \propto \omega^{d-1}$ , with the higher-frequency vibrational density of states for the fractons,  $N_{\text{fr}}(\omega) \propto \omega^{\bar{d}-1}$ . A smoothness assumption for the force constant and mass,

as one changes the length scale through  $\xi_p$  (equivalent to increasing  $\omega$  through  $\omega_c$ ) would lead to<sup>(8)</sup>  $N_{fr}(\omega_c)/N_{ph}(\omega_c) = d/d_c$ , or a sharp drop in  $N(\omega)$  at crossover. This is contradictory to experiment.<sup>(9,16)</sup>

This difference in behavior was puzzling until very recent work by Derrida *et al.*<sup>(29)</sup> They used the EMA treatment of diffusion for bond percolation by Webman<sup>(12)</sup> and Odagaki and Lax,<sup>(30)</sup> and performed the same trick as described in the previous section to obtain  $N(\omega)$ . We reproduce their curve for  $N(\omega)$  at  $d = 3$  in Fig. 1. Their result is both surprising and gratifying. The two regimes, phonon and fracton, are clearly seen. The slope of  $N(\omega)$  in the low-frequency regime is 2, as expected for phonons at  $d = 3$ . The magnitude of  $N(\omega)$  scales with a velocity of sound proportional to  $(p - p_c)^{1/2}$ . At a crossover frequency

$$\omega_c = [d^2/2(d-1)\psi_d]^{1/2}(p - p_c) \quad (11)$$

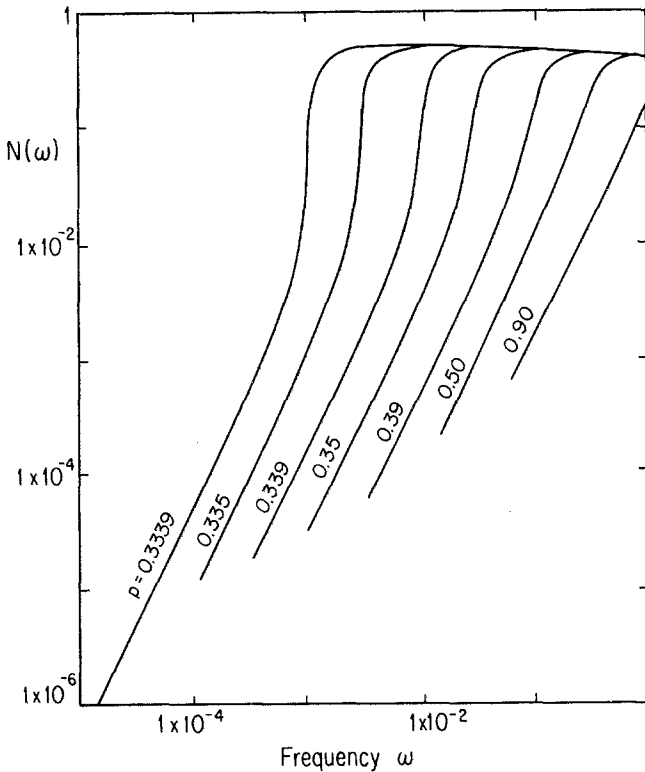


Fig. 1. The vibrational density of states for the simple cubic lattice percolating network for  $d = 3$ , as a function of frequency  $\omega$ , calculated within the effective medium approximation for different bond concentrations  $p$ . Within EMA,  $p_c = 1/3 = 0.3333\dots$

where  $\psi_d$  is defined in Ref. 29, and is of order unity, a rapid rise in  $N(\omega)$  occurs to the fracton value. The latter is a constant, implying  $\bar{d}_{\text{EMA}} = 1$ . Their calculations show that in fact  $\bar{d}_{\text{EMA}} = 1$  for  $2 \leq d \leq 4$ , consistent with the Alexander–Orbach conjecture relative to the constancy of  $\bar{d}$ . The actual EMA value is not accurate (unity instead of  $4/3$ ). This is not uncommon for EMA: it often gives the correct physical behavior, but with values for the exponents which differ from the correct values. The magnitude of the rapidly rising portion of  $N(\omega)$  accentuates as  $p \rightarrow p_c$ , varying as  $(p - p_c)^{1-(d/2)}$ . This is clearly seen in Fig. 1 as one moves away from  $p_c$  ( $p_c = 1/d$  within EMA).

The principal EMA result, the rapid rise in  $N(\omega)$  at  $\omega_c$ , agrees well with the Kelham and Rosenberg<sup>(9)</sup> analysis of the specific heat of the epoxy-resins. It is also in striking agreement with the structure often observed in the far infrared absorption of glasses and *a*-Ge. We reproduce in Figs. 2a and 2b the Fig. 1 of Refs. 10 and 11, respectively. One sees a “knee” in the absorption curves for the glasses  $\text{GeO}_2$ ,  $\text{B}_2\text{O}_3$ , and  $\text{SiO}_2$  in Fig. 2a. This is not unlike the EMA shape exhibited in Fig. 1 if  $\bar{d}$  were greater than unity ( $4/3?$ ). That is, there does appear to be a significant decrease in slope for the optical absorption (presumably proportional to the vibrational density of states) at frequencies above (roughly)  $20 \text{ cm}^{-1}$  for  $\text{GeO}_2$ ,  $15 \text{ cm}^{-1}$  for  $\text{B}_2\text{O}_3$ , and  $50 \text{ cm}^{-1}$  for  $\text{SiO}_2$  glasses. There even appears to be evidence for a rapidly rising portion, of the same general shape as for the EMA result for  $N(\omega)$  in Fig. 1. The case of pure *a*-Ge is less clear, but Fig. 2b does exhibit a sudden decrease in slope at roughly  $40 \text{ cm}^{-1}$ . Note also that no structure appears present in the crystalline material at the same frequency.

It is certainly not obvious why amorphous structures should exhibit dynamical properties analogous to percolating networks. Nevertheless, the similarities are striking (see Ref. 8 for a fuller discussion). Again, neutron diffraction measurements of  $N(\omega)$  in the frequency range between 10 and  $50 \text{ cm}^{-1}$  on amorphous materials would be most interesting.

In an effort to search for the physical origin of the rapid rise in  $N(\omega)$  at  $\omega_c$ , we compute now the diffusion constant within EMA. We have already calculated  $\langle r^2(t) \rangle$  analytically within EMA in Ref. 29. Using the relation<sup>(17)</sup>  $D(t) = \frac{1}{2} d \langle r^2(t) \rangle / dt$ , we find

$$D(t) = \frac{d^2(p - p_c)}{(d - 1)} \left\{ \frac{1}{2} + \left( \frac{1}{\pi} \right)^{1/2} \left[ \int_0^{x^{1/2}} e^{-u^2} du + \left[ \frac{1}{2(x)^{1/2}} \right] x^{-x} \right] \right\} \quad (12)$$

where  $x = t/t_0$ , and<sup>(31)</sup>  $t_0 = 2(d - 1)\psi_d/d^2(p - p_c)^2$ . We plot  $D(t)$  vs.  $t/t_0$  in Fig. 3. It is seen that no structure is present in the crossover regime (near  $x = t/t_0 = 1$ ). The EMA expression for the diffusion constant is proportional to  $t^{-1/2}$  at short times, a result first obtained by Webman.<sup>(12)</sup> As  $t/t_0$

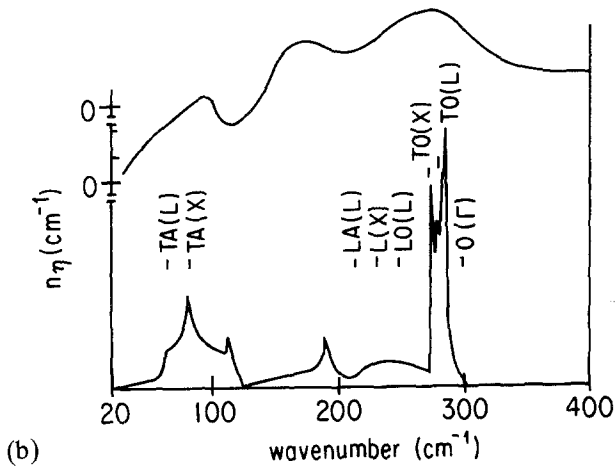
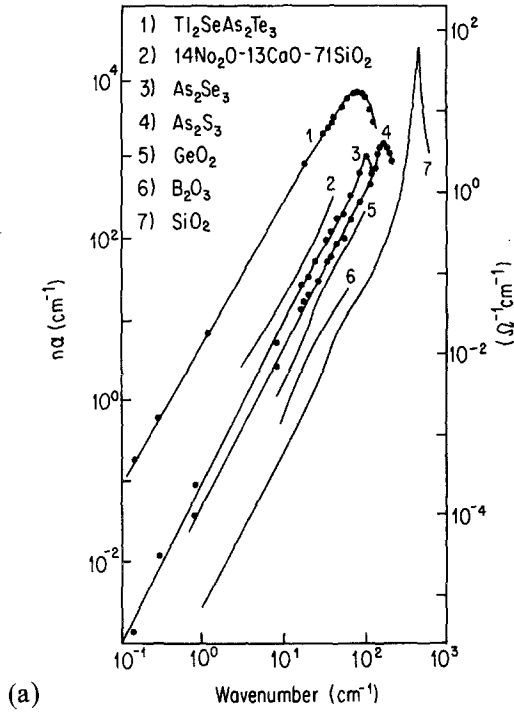


Fig. 2. (a) Room temperature, far infrared, and microwave conductivity  $\sigma$  (or index of refraction times absorption coefficient,  $n\alpha$ ) of selected amorphous materials as a function of wave number (or frequency) on a log-log scale. Redrawn from Fig. 1 of Ref. 10. (b) The product of  $n\eta$  ( $\text{cm}^{-1}$ ) vs. wave number ( $\text{cm}^{-1}$ ) for pure  $a$ -Ge. Here,  $n$  is the refractive index and  $\eta$  the absorption coefficient. Included are the crystalline density of states for Ge and the position of singularities arising from critical points at  $\Gamma$ ,  $X$ , and  $L$  of the Brillouin zone. Redrawn from Fig. 1 of Ref. 11.



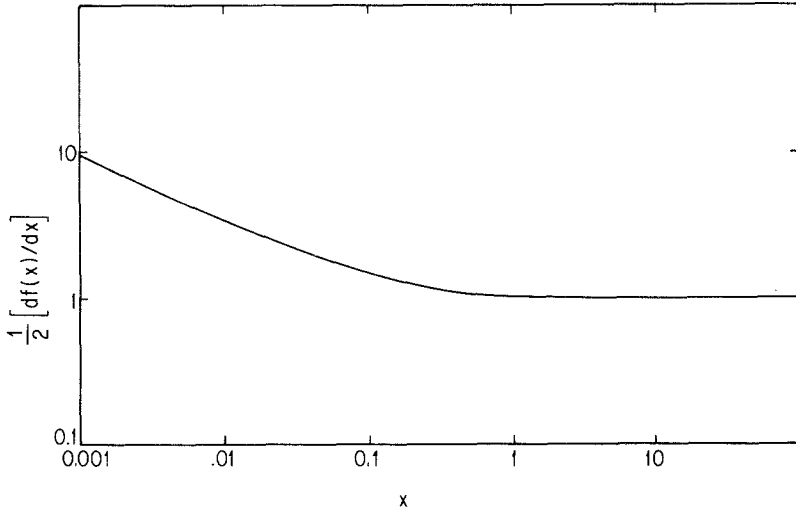


Fig. 3. A plot of the diffusion constant  $D(t)/[d^2(p - p_c)/(d - 1)]$ , from Eq. (12) in the text. The term in curly brackets in Eq. (12) equals  $(\frac{1}{2}[df(x)/dx])$ .

increases towards unity,  $D(t)$  crosses over to a constant, as expected for the Euclidean regime.

Thus, within the same (EMA) approximation,  $N(\omega)$  exhibits a rapid rise at crossover, whereas  $D(t)$  behaves quite smoothly. The reasons for this difference will be explored in detail in the following section.

#### 4. VIBRATIONAL DISPERSION RELATION

Figure 1 of the preceding section exhibits strange shapes for the vibrational density of states on a percolating network. We examine the vibrational eigenstates in more detail in this section in order to understand the physical reasons behind the rapid rise in  $N(\omega)$  near  $\omega_c$ . A full account of this work can be found in the paper by Entin-Wohlman *et al.*<sup>(31)</sup>

We work in the EMA where the randomly occupied bonds are replaced by an “effective” bond of strength  $\bar{W}$  which, however, is frequency dependent:  $\bar{W}[\omega(q)]$ , where  $\omega(q)$  is the  $q$ -dependent frequency.  $\bar{W}$  itself satisfies a self-consistency equation for which we refer the reader to Ref. 31 for full details.

The dispersion relations are found from the poles of the Green’s function. They take the form

$$\omega^2(q) = q^2 \bar{W}[\omega(q)] \tag{13}$$

$\bar{W}$  is complex, so we denote (for  $q$  real),

$$\omega(q) = \omega_q - i\delta_q \quad (14)$$

where  $\omega_q$  and  $\delta_q$  denote the wave-vector-dependent frequency and decay rate of the wavelike excitations, respectively. The full equations for Eq. (13) are complicated, so we do not exhibit them here. We do plot the solution of Eq. (13) for  $p$  near  $p_c$  in Fig. 4. The various interesting regions of  $q$  are examined below.

#### 4.1. $q$ Small

We find ( $p > p_c$ ),

$$\omega_q^2 \sim [(p - p_c)/(1 - p_c)]q^2 \quad (15a)$$

$$\delta_q \sim \omega_q[q^d/(p - p_c)] \quad (15b)$$

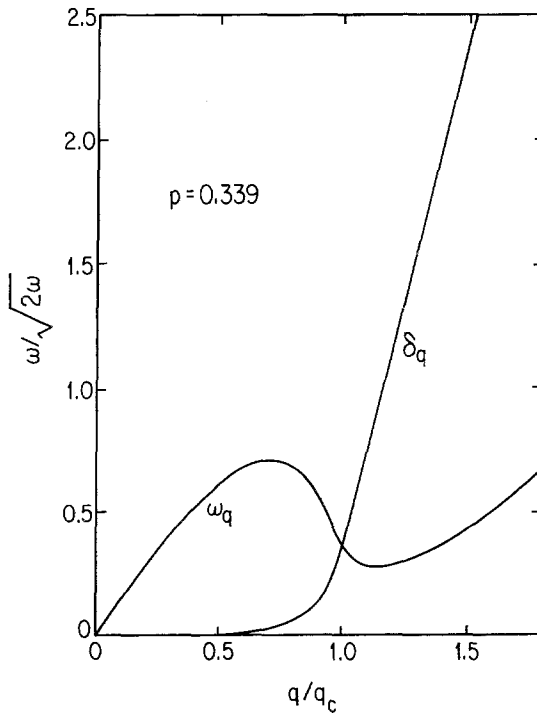


Fig. 4. A plot of the real and imaginary parts of  $\omega$ ,  $\omega_q$ , and  $\delta_q$ , respectively, vs.  $q$  for  $d = 3$ . Here,  $p = 0.339$  and  $\omega_c = 0.0104$ . The critical percolation concentration  $p_c = 1/3$ .

That is, the dispersion law for small  $q$  (large length scale) is that of phonons,  $\omega_q \sim c_s q$ , with a sound velocity  $c_s$  scaling as  $(p - p_c)^{1/2}$ . The decay rate in this limit is much smaller than  $\omega_q$  so that the excitations are well defined. However, this holds only for  $p > p_c$  and for small  $q$  such that

$$q^2 < q_c^2, \quad q_c^2 = 2(p - p_c)/p_c \psi_d \tag{16}$$

**4.2.  $q$  in the Vicinity of  $q_c$**

From Fig. 4, we see that  $\omega_c$  as defined by Eq. (11) represents the maximum value of  $\omega$  in the phonon regime (i.e., for  $q < q_c$ ). Further, the decay rate [i.e., the imaginary part of  $\omega(q)$ ] increases remarkably fast. For  $q > q_c$ ,  $\delta_q \gg \omega_q$ . This means that the poles of the propagator, Eq. (13), do not lie close to the real axis, and hence do not describe a physically accessible excitation spectrum. The frequency  $\omega_c$  was shown in Section 3 to mark the crossover between the phonon and fracton portions of the vibrational density of states, within the EMA. Our calculation shows that at frequencies less than  $\omega_c$ , or  $q < q_c$ , the plane wave representation which yields a dispersion law of  $\omega = c_s q$  is indeed meaningful. However, as  $q$  increases beyond  $q_c$ , the decay rate  $\delta_q$  increases sharply. Consequently, the plane wave representation is no longer adequate. We note that this breakdown of the plane wave representation occurs at  $q$  of the order of  $q_c$ , which itself scales as  $(p - p_c)^{1/2}$ . As shown in Ref. 29, this is related within EMA to the excitation length scale equaling the connectivity length of the percolation problem, in accordance with the conclusions of Ref. 2. That is,  $1/q_c \sim \xi_p$ , the percolation correlation length. Hence,  $\xi_p$  sets the length scale. For  $1/q > \xi_p$ , one is in the phonon regime. For  $1/q < \xi_p$ , one is in the fracton regime.

**4.3.  $q$  Large**

It is clear that  $q$  is not a useful representation in this regime. Rather, we recognize that a “damping length”  $l$  can be associated with

$$1/2l(\omega) = |\text{Im}[\omega/(\bar{W}(\omega))^{1/2}]| \tag{17}$$

from Eq. (13). From Ref. 31, this leads to the relationship

$$\omega = \omega_c + [\psi_d p_c / 8(1 - p_c)]^{1/2} (1/l^2) \tag{18}$$

It is interesting to note that the scaling arguments of Ref. 2 lead to the relation within the fracton regime,

$$\omega_{fr}(l) \propto l^{-\bar{d}/\bar{d}} \tag{19}$$

Noting that  $\bar{d}_{\text{EMA}} = 2$ ,  $\bar{\bar{d}}_{\text{EMA}} = 1$  at  $d = 3$ , we see that the explicit EMA fracton dispersion law is in agreement with the scaling form if one utilizes the EMA values for  $\bar{d}$  and  $\bar{\bar{d}}$ .

We are now in a position to analyze the density of states as exhibited in Fig. 1. One sees that the sudden rise at  $\omega_c$  is associated with the "flat" dispersion regime at  $\omega_c$ . It is the near vanishing of  $d\omega/dq$  near  $\omega_c$  which generates the rapid rise in  $N(\omega)$  at  $\omega_c$ . The finite imaginary part  $\delta_q$  causes the pole of the propagator to lie off the real axis, thereby avoiding a divergence at  $\omega_c$ . The closer one is to  $p_c$ , the smaller is  $\delta_q$  at  $\omega_c$ , and the more rapid the rise in  $N(\omega)$  at  $\omega_c$ .

Neutron diffraction experiments to measure the dispersion law of amorphous materials will be difficult because of the rapid rise in  $\delta_q$  with  $q$ . As a first try, however, it would be useful to see if (at the least)  $\delta_q$  were found to increase according to Eq. (15b). We wish to emphasize that there is no lifetime damping of the phonons or fractons in this calculation: rather, it is a wave vector damping associated with the failure of a plane wave description. Fixing the momentum transfer, as one does in a neutron diffraction experiment, causes a breadth in the measurement of  $\omega_q$ . It would be of great interest to follow the predictions of Eq. (15) as far as one can into the  $\omega, q$  plane before the damping becomes excessive.

## 5. SUMMARY

The preceding sections have explored the three dimensions which appear to be required for a full description of fracton dynamics,  $d, \bar{d}, \bar{\bar{d}}$ . The percolating network was chosen as a "laboratory" to examine not only an explicit fracton excitation spectrum, but also the crossover from Euclidean to fractal space. The conjecture<sup>(2)</sup> that  $\bar{\bar{d}} = 4/3$  now appears to be off by  $\sim 2\%$  at  $d = 2$ ,<sup>(24,25)</sup> but may be exact for  $d$  such that  $\bar{d} \geq 2$ .<sup>(26)</sup>

The vibrational density of states crosses over from  $\omega^{d-1}$  to  $\omega^{\bar{\bar{d}}-1}$  as one crosses over from Euclidean to fractal space, with a concomitant crossover in the excitation spectrum from  $\omega \sim c, q$  to  $\omega \propto l^{-\bar{\bar{d}}/\bar{d}}$  where  $l$  represents the length scale of the fracton excitation. For  $\bar{\bar{d}} < 2$ , Rammal and Toulouse<sup>(3)</sup> have shown that the fractons are localized spatially, so that such a description of the fracton regime is appropriate.

We introduced an effective medium approximation in order to more fully explore the crossover properties of the density of states and the dispersion law. Remarkably, we discovered<sup>(29)</sup> a sudden rise in  $N(\omega)$  at the crossover frequency  $\omega_c$ . This was shown in Section 4 to be caused by the peculiar ("roton"-like, but highly damped) flattening off of the dispersion curve at  $\omega_c$ , with a subsequent drop and rise in  $\omega(q)$  as one passes into the fracton regime.<sup>(1)</sup>

These results are suggestive of an explanation for the nearly universal structure of the specific heat and thermal conductivity of amorphous materials.<sup>(9,16)</sup> All seem to exhibit an  $N(\omega)$  which rises steeply at relatively low energies (e.g.,  $\sim 8\text{--}12$  K for the epoxy-resins), and a plateau in the thermal conductivity in roughly the equivalent temperature regime. The localization of the fracton excitations<sup>(3)</sup> can be regarded as an explanation for the near-zero phonon mean free path above the same low energy, as extracted from conventional analysis for nearly all amorphous materials.

We are unable to explain why amorphous materials should exhibit fracton excitations in their vibrational spectra. Such behavior requires the existence of a characteristic length scale, below which the structure participating in the dynamics is self-similar. At present it appears difficult to explore the length scale issue directly by other means.

Optical infrared absorption in amorphous structures is another technique for probing  $N(\omega)$ . "Impurity-induced" absorption is maximal so that the optical absorption profile should mirror  $N(\omega)$ . Where experimental results are available, structure appears to be present which is not inconsistent with the EMA predictions for  $N(\omega)$  on a percolating network.<sup>(10,11)</sup>

It is possible to apply the EMA procedures to percolating ferro- and antiferromagnets. The former follows directly from the vibrational solutions because the eigenvalue equation is identical (with the exception of the spectral parameter).<sup>(33)</sup> The latter requires a separate EMA calculation which has recently been carried out by Yu and Orbach.<sup>(34)</sup> They find that the rapid rise in  $N(\omega)$  near  $\omega_c$  for lattice vibrations is absent for antiferromagnetic spin excitations. Rather, only a rather smooth crossover takes place between the long-length-scale magnon and short-length-scale fraction regimes. This distinctively different behavior should be found in neutron diffraction experiments which measure  $N(\omega)$  in the vicinity of  $\omega_c$ . Such experiments are easier than for lattice vibrations because  $\omega_c$  scales as  $(p - p_c)^2$  for magnetic excitations. Thus, one need not have to work so close to  $p_c$  for magnetic materials as for vibrational spectra in order to observe the interesting features of  $N(\omega)$ .

In conclusion, we have tried to present a description of the physics of fractal excitations, both from a scaling and an EMA calculation for the percolating network. Our results are applicable to vibrational and spin-wave excitations, as well as for a certain class of tight binding electrons. We await specific experimental tests of our predictions.

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